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## Appendix: Proof of Proposition 3

The problem is as follows.

$$\max_{k_t}\left\{\sum_{t=1}^{\infty}\beta^t u\left(c_t\right)\right\}$$
$$s.t.\left(1-d\right)k_{t-1}+\left(1-X\right)f\left(k_{t-1}\right)+\lambda_{t-1}Xf\left(k_{t-1}\right)=c_t+k_t+\lambda_t\left(1+\theta\right)f\left(k_t\right)EX$$
$$for\ t=1,2,3\cdots$$
$$k_0,\lambda_0\ are\ given.$$

The existence of the optimal path has been proved in Sutherland (1970). Here we focus on the proof of the convergence of the path. In the proof, we need the following Lemma.

Lemma: (Fuente, 2000, Theorem 10.3.4) Let $\overline{x}$ represent the steady state of the discrete system $x_{t+1}=g\left(x_t\right)$,

(1) $\overline{x}$ is a locally steady state if all the eigenvalues of $Dg\left(\overline{x}\right)$ are less than 1 in absolute value.

(2) $\overline{x}$ is a unsteady state if at least one of the eigenvalues is greater than 1 in absolute value.

(3) $\overline{x}$ is steady or unsteady if one of the eigenvalues equals 1 and the other are less than 1 in absolute value. At this time, the Jacobian matrix test is inconclusive.

The Bellman equation of the model is,

$$v\left(k_t,\lambda_t\right)=\max_{k_{t+1},\lambda_{t+1}}\left\{\begin{array}{l}Eu\left[\left(1-d\right)k_t+\left(1-X\right)f\left(k_t\right)+\lambda_tXf\left(k_t\right)-k_{t+1}-\lambda_{t+1}f\left(k_{t+1}\right)P\right]\\+\beta v\left(k_{t+1},\lambda_{t+1}\right)\end{array}\right\}$$

Then, $k_{t+1}$, $\lambda_{t+1}$ satisfies

$$\left[-1-\lambda_{t+1}f'\left(k_{t+1}\right)P\right]Eu'+\beta\frac{\partial v\left(k_{t+1},\lambda_{t+1}\right)}{\partial k_{t+1}}=0 \quad \text{(A1)}$$

$$-f\left(k_{t+1}\right)PEu'+\beta\frac{\partial v\left(k_{t+1},\lambda_{t+1}\right)}{\partial\lambda_{t+1}}=0 \quad \text{(A2)}$$

Let $k_{t+1}=g\left(k_t,\lambda_t\right),\lambda_{t+1}=h\left(k_t,\lambda_t\right)$ be the optimal path (Sutherland, 1970). Based on the Lemma, we only need to prove that all the eigenvalues of the Jacobian matrix are less than 1 in absolute value.

**The first task is to derive out the Jacoby matrix.** Differentiating both sides of equation (A1) with respect to $k_t,\lambda_t$, we have

$$\begin{array}{l}\left[-\frac{\partial h}{\partial k_t}f'\left(k_{t+1}\right)P-\lambda_{t+1}f''\left(k_{t+1}\right)\frac{\partial h}{\partial k_t}P\right]Eu'+\left[-1-\lambda_{t+1}f'\left(k_{t+1}\right)P\right]\times\\E\left\{\left[\left(1-d\right)+\left(1-X\right)f'\left(k_t\right)+\lambda_tXf'\left(k_t\right)-\frac{\partial g}{\partial k_t}-\frac{\partial h}{\partial k_t}f\left(k_{t+1}\right)P-\lambda_{t+1}f'\left(k_{t+1}\right)\frac{\partial g}{\partial k_t}P\right]u''\right\} \quad \text{(A3)}\\+\beta\frac{\partial^2 v\left(k_{t+1},\lambda_{t+1}\right)}{\partial k_{t+1}^2}\frac{\partial g}{\partial k_t}+\beta\frac{\partial^2 v\left(k_{t+1},\lambda_{t+1}\right)}{\partial k_{t+1}\partial\lambda_{t+1}}\frac{\partial h}{\partial k_t}=0\end{array}$$

$$\left[-\frac{\partial h}{\partial \lambda_t} f'(k_{t+1})P-\lambda_{t+1} f''(k_{t+1})\frac{\partial h}{\partial \lambda_t}P\right]Eu'+\left[-1-\lambda_{t+1} f'(k_{t+1})P\right]\times$$
$$E\left\{\left[Xf(k_t)-\frac{\partial g}{\partial \lambda_t}-\frac{\partial h}{\partial \lambda_t} f(k_{t+1})P-\lambda_{t+1} f'(k_{t+1})\frac{\partial g}{\partial \lambda_t}P\right]u''\right\} \quad (A4)$$
$$+\beta\frac{\partial^2 v(k_{t+1},\lambda_{t+1})}{\partial k_{t+1}^2}\frac{\partial g}{\partial \lambda_t}+\beta\frac{\partial^2 v(k_{t+1},\lambda_{t+1})}{\partial k_{t+1}\partial \lambda_{t+1}}\frac{\partial h}{\partial \lambda_t}=0$$

Differentiating both sides of equation (A2) with respect to $k_t, \lambda_t$, we have

$$-f'(k_{t+1})\frac{\partial k_{t+1}}{\partial k_t}PEu'$$
$$-E\left\{\left[(1-d)+(1-X)f'(k_t)+\lambda_t Xf'(k_t)-\frac{\partial g}{\partial k_t}-\frac{\partial h}{\partial k_t} f(k_{t+1})P-\lambda_{t+1} f'(k_{t+1})\frac{\partial g}{\partial k_t}P\right]f(k_{t+1})Pu''\right\} \quad (A5)$$
$$+\beta\frac{\partial v^2(k_{t+1},\lambda_{t+1})}{\partial k_{t+1}\partial \lambda_{t+1}}\frac{\partial g}{\partial k_t}+\beta\frac{\partial v^2(k_{t+1},\lambda_{t+1})}{\partial \lambda_{t+1}^2}\frac{\partial h}{\partial k_t}=0$$

$$-f'(k_{t+1})\frac{\partial g}{\partial \lambda_t}PEu'$$
$$-E\left\{\left[Xf(k_t)-\frac{\partial g}{\partial \lambda_t}-\frac{\partial h}{\partial \lambda_t} f(k_{t+1})P-\lambda_{t+1} f'(k_{t+1})\frac{\partial g}{\partial \lambda_t}P\right]f(k_{t+1})Pu''\right\} \quad (A6)$$
$$+\beta\frac{\partial v^2(k_{t+1},\lambda_{t+1})}{\partial k_{t+1}\partial \lambda_{t+1}}\frac{\partial g}{\partial \lambda_t}+\beta\frac{\partial v^2(k_{t+1},\lambda_{t+1})}{\partial \lambda_{t+1}^2}\frac{\partial h}{\partial \lambda_t}=0$$

Writing equation (A3)- (A6) into the form of matrix,

$$\begin{pmatrix} \left[-\frac{\partial h}{\partial k_t} f'(k_{t+1})P-\lambda_{t+1} f''(k_{t+1})\frac{\partial g}{\partial k_t}P\right]Eu' & -f'(k_{t+1})\frac{\partial g}{\partial k_t}PEu' \\ \left[-\frac{\partial h}{\partial \lambda_t} f'(k_{t+1})P-\lambda_{t+1} f''(k_{t+1})\frac{\partial g}{\partial \lambda_t}P\right]Eu' & -f'(k_{t+1})\frac{\partial g}{\partial \lambda_t}PEu' \end{pmatrix}$$
$$+\begin{pmatrix} \left[-1-\lambda_{t+1} f'(k_{t+1})P\right]E\left\{\left[(1-d)+f'(k_t)-(1-\lambda_t)Xf'(k_t)-\frac{\partial g}{\partial k_t}-\frac{\partial h}{\partial k_t} f(k_{t+1})P-\lambda_{t+1} f'(k_{t+1})\frac{\partial g}{\partial k_t}P\right]u''\right\} & -E\left\{\left[(1-d)+f'(k_t)-(1-\lambda_t)Xf'(k_t)-\frac{\partial g}{\partial k_t}-\frac{\partial h}{\partial k_t} f(k_{t+1})P-\lambda_{t+1} f'(k_{t+1})\frac{\partial g}{\partial k_t}P\right]f(k_{t+1})Pu''\right\} \\ \left[-1-\lambda_{t+1} f'(k_{t+1})P\right]E\left\{\left[Xf(k_t)-\frac{\partial g}{\partial \lambda_t}-\frac{\partial h}{\partial \lambda_t} f(k_{t+1})P-\lambda_{t+1} f'(k_{t+1})\frac{\partial g}{\partial \lambda_t}P\right]u''\right\} & -E\left\{\left[Xf(k_t)-\frac{\partial g}{\partial \lambda_t}-\frac{\partial h}{\partial \lambda_t} f(k_{t+1})P-\lambda_{t+1} f'(k_{t+1})\frac{\partial g}{\partial \lambda_t}P\right]f(k_{t+1})Pu''\right\} \end{pmatrix}$$
$$+\begin{pmatrix} \beta\frac{\partial^2 v(k_{t+1},\lambda_{t+1})}{\partial k_{t+1}^2}\frac{\partial g}{\partial k_t}+\beta\frac{\partial^2 v(k_{t+1},\lambda_{t+1})}{\partial k_{t+1}\partial \lambda_{t+1}}\frac{\partial h}{\partial k_t} & \beta\frac{\partial v^2(k_{t+1},\lambda_{t+1})}{\partial k_{t+1}\partial \lambda_{t+1}}\frac{\partial g}{\partial k_t}+\beta\frac{\partial v^2(k_{t+1},\lambda_{t+1})}{\partial \lambda_{t+1}^2}\frac{\partial h}{\partial k_t} \\ \beta\frac{\partial^2 v(k_{t+1},\lambda_{t+1})}{\partial k_{t+1}^2}\frac{\partial g}{\partial \lambda_t}+\beta\frac{\partial^2 v(k_{t+1},\lambda_{t+1})}{\partial k_{t+1}\partial \lambda_{t+1}}\frac{\partial h}{\partial \lambda_t} & \beta\frac{\partial v^2(k_{t+1},\lambda_{t+1})}{\partial k_{t+1}\partial \lambda_{t+1}}\frac{\partial g}{\partial \lambda_t}+\beta\frac{\partial v^2(k_{t+1},\lambda_{t+1})}{\partial \lambda_{t+1}^2}\frac{\partial h}{\partial \lambda_t} \end{pmatrix}=0$$

(A7)

That is,

$$\begin{pmatrix} -f'(k_{t+1})PEu'\frac{\partial h}{\partial k_t} & -f'(k_{t+1})PEu'\frac{\partial g}{\partial k_t} \\ -f'(k_{t+1})PEu'\frac{\partial h}{\partial \lambda_t} & -f'(k_{t+1})PEu'\frac{\partial g}{\partial \lambda_t} \end{pmatrix}$$

$$+\begin{pmatrix} \left[-1-\lambda_{t+1}f'(k_{t+1})P\right]E\left\{\left[(1-d)+f'(k_t)-(1-\lambda_t)Xf'(k_t)-\frac{\partial g}{\partial k_t}-\frac{\partial h}{\partial k_t}f(k_{t+1})P-\lambda_{t+1}f'(k_{t+1})\frac{\partial g}{\partial k_t}P\right]u''\right\} & -E\left\{\left[(1-d)+f'(k_t)-(1-\lambda_t)Xf'(k_t)-\frac{\partial g}{\partial k_t}-\frac{\partial h}{\partial k_t}f(k_{t+1})P-\lambda_{t+1}f'(k_{t+1})\frac{\partial g}{\partial k_t}P\right]f(k_{t+1})Pu''\right\} \\ \left[-1-\lambda_{t+1}f'(k_{t+1})P\right]E\left\{\left[Xf(k_t)-\frac{\partial g}{\partial \lambda_t}-\frac{\partial h}{\partial \lambda_t}f(k_{t+1})P-\lambda_{t+1}f'(k_{t+1})\frac{\partial g}{\partial \lambda_t}P\right]u''\right\} & -E\left\{\left[Xf(k_t)-\frac{\partial g}{\partial \lambda_t}-\frac{\partial h}{\partial \lambda_t}f(k_{t+1})P-\lambda_{t+1}f'(k_{t+1})\frac{\partial g}{\partial \lambda_t}P\right]f(k_{t+1})Pu''\right\} \end{pmatrix}$$

$$+\begin{pmatrix} \beta\frac{\partial^2 v(k_{t+1},\lambda_{t+1})}{\partial k_{t+1}^2}\frac{\partial g}{\partial k_t}+\beta\frac{\partial^2 v(k_{t+1},\lambda_{t+1})}{\partial k_{t+1}\partial\lambda_{t+1}}\frac{\partial h}{\partial k_t} & \beta\frac{\partial v^2(k_{t+1},\lambda_{t+1})}{\partial k_{t+1}\partial\lambda_{t+1}}\frac{\partial g}{\partial k_t}+\beta\frac{\partial v^2(k_{t+1},\lambda_{t+1})}{\partial \lambda_{t+1}^2}\frac{\partial h}{\partial k_t} \\ \beta\frac{\partial^2 v(k_{t+1},\lambda_{t+1})}{\partial k_{t+1}^2}\frac{\partial g}{\partial \lambda_t}+\beta\frac{\partial^2 v(k_{t+1},\lambda_{t+1})}{\partial k_{t+1}\partial\lambda_{t+1}}\frac{\partial h}{\partial \lambda_t} & \beta\frac{\partial v^2(k_{t+1},\lambda_{t+1})}{\partial k_{t+1}\partial\lambda_{t+1}}\frac{\partial g}{\partial \lambda_t}+\beta\frac{\partial v^2(k_{t+1},\lambda_{t+1})}{\partial \lambda_{t+1}^2}\frac{\partial h}{\partial \lambda_t} \end{pmatrix}=0 \tag{A8}$$

Deducting the second derivative of the Bellman equation, we have,

$$\frac{\partial^2 v(k_t,\lambda_t)}{\partial k_t^2}=\max_{k_{t+1},\lambda_{t+1}} E\left\{\left[(1-d)+f'(k_t)-(1-\lambda_t)Xf'(k_t)\right]^2 u''\right\}$$

$$\frac{\partial^2 v(k_t,\lambda_t)}{\partial \lambda_t^2}=\max_{k_{t+1},\lambda_{t+1}} E\left\{f^2(k_t)X^2u''\right\}$$

$$\frac{\partial^2 v(k_t,\lambda_t)}{\partial k_t\partial\lambda_t}=\max_{k_{t+1},\lambda_{t+1}} E\left\{f'(k_t)Xu'+Xf(k_t)\left[(1-d)+f'(k_t)-(1-\lambda_t)Xf'(k_t)\right]u''\right\}$$

Let $m=(1-d)+f'(k)-(1-\lambda)Xf'(k)$, at the point satisfying $f'(k)=\left[1-(1-d)\beta\right]/(\beta-P)$ and $P=\beta\frac{E(Xu')}{Eu'}$, equation (A8) is written as

$$\begin{pmatrix} \frac{\partial g}{\partial k_t} & \frac{\partial h}{\partial k_t} \\ \frac{\partial g}{\partial \lambda_t} & \frac{\partial h}{\partial \lambda_t} \end{pmatrix}\begin{pmatrix} \left[1+\lambda f'(k)P\right]^2 Eu''+\beta E(m^2u'') & \left[1+\lambda f'(k)P\right]f(k)PEu''+\beta f(k_t)E(Xmu'') \\ \left[1+\lambda f'(k)P\right]f(k)PEu''+\beta f(k)E(Xmu'') & f^2(k)P^2Eu''+\beta f^2(k)E(X^2u'') \end{pmatrix} \tag{A9}$$

$$=\begin{pmatrix} \left[1+\lambda f'(k)P\right]E(mu'') & f(k)PE(mu'') \\ \left[1+\lambda f'(k)P\right]f(k)E(Xu'') & f^2(k)PE(Xu'') \end{pmatrix}$$

Let $Q=\left\{\left[1+\lambda f'(k)P\right]^2 Eu''+\beta E(m^2u'')\right\}\left\{f^2(k)P^2Eu''+\beta f^2(k)E(X^2u'')\right\}-\left\{\left[1+\lambda f'(k)P\right]f(k)PEu''+\beta f(k_t)E(Xmu'')\right\}^2$, we have

$$\begin{pmatrix} \frac{\partial g}{\partial k_t} & \frac{\partial h}{\partial k_t} \\ \frac{\partial g}{\partial \lambda_t} & \frac{\partial h}{\partial \lambda_t} \end{pmatrix}=\frac{1}{Q}\begin{pmatrix} \left[1+\lambda f'(k)P\right]E(mu'') & f(k)PE(mu'') \\ \left[1+\lambda f'(k)P\right]f(k)E(Xu'') & f^2(k)PE(Xu'') \end{pmatrix} \tag{A10}$$

$$\begin{pmatrix} f^2(k)P^2Eu''+\beta f^2(k)E(X^2u'') & -\left\{\left[1+\lambda f'(k)P\right]f(k)PEu''+\beta f(k_t)E(Xmu'')\right\} \\ -\left\{\left[1+\lambda f'(k)P\right]f(k)PEu''+\beta f(k_t)E(Xmu'')\right\} & \left[1+\lambda f'(k)P\right]^2 Eu''+\beta E(m^2u'') \end{pmatrix}$$

Assume that the Jacoby matrix $\begin{pmatrix} \dfrac{\partial g}{\partial k_t} & \dfrac{\partial h}{\partial k_t} \\ \dfrac{\partial g}{\partial \lambda_t} & \dfrac{\partial h}{\partial \lambda_t} \end{pmatrix} = \dfrac{1}{Q}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, therefore

$$A_{11} = \left\{\left[1+\lambda f'(k)P\right]E\left(X^2u''\right)-PE\left(Xmu''\right)\right\}\beta f^2(k)E\left(mu''\right)$$

$$A_{12} = \left\{\left[1+\lambda f'(k)P\right]E\left(X^2u''\right)-PE\left(Xmu''\right)\right\}\beta f^3(k)E\left(Xu''\right)$$

$$A_{21} = \left\{PE\left(m^2u''\right)-\left[1+\lambda f'(k)P\right]E\left(Xmu''\right)\right\}\beta f(k)E\left(mu''\right)$$

$$A_{22} = \left\{PE\left(m^2u''\right)-\left[1+\lambda f'(k)P\right]E\left(Xmu''\right)\right\}\beta f^2(k)E\left(Xu''\right)$$

Setting $U=\left\{\left[1+\lambda f'(k)P\right]E\left(X^2u''\right)-PE\left(Xmu''\right)\right\}$ and $V=\left\{PE\left(m^2u''\right)-\left[1+\lambda f'(k)P\right]E\left(Xmu''\right)\right\}$, the Jacobian matrix becomes

$$\begin{pmatrix} \dfrac{\partial g}{\partial k_t} & \dfrac{\partial h}{\partial k_t} \\ \dfrac{\partial g}{\partial \lambda_t} & \dfrac{\partial h}{\partial \lambda_t} \end{pmatrix} = \frac{1}{Q}\begin{pmatrix} U\beta f^2(k)E\left(mu''\right) & V\beta f(k)E\left(mu''\right) \\ U\beta f^3(k)E\left(Xu''\right) & V\beta f^2(k)E\left(Xu''\right) \end{pmatrix} \tag{A11}$$

**Then we prove that both of the eigenvalues are less than 1 in absolute value.** Assume $\xi$ representing the eigenvalues of the Jacobian matrix, that is

$$\begin{vmatrix} \xi-\dfrac{1}{Q}U\beta f^2(k)E\left(mu''\right) & -\dfrac{1}{Q}V\beta f(k)E\left(mu''\right) \\ -\dfrac{1}{Q}U\beta f^3(k)E\left(Xu''\right) & \xi-\dfrac{1}{Q}V\beta f^2(k)E\left(Xu''\right) \end{vmatrix} = 0$$

i.e.,

$$\left[\xi-\frac{1}{Q}U\beta f^2(k)E\left(mu''\right)\right]\left[\xi-\frac{1}{Q}V\beta f^2(k)E\left(Xu''\right)\right]-\frac{1}{Q^2}U\beta f^3(k)E\left(Xu''\right)V\beta f(k)E\left(mu''\right)=0 \tag{A12}$$

We have

$$\xi^2-\left[\frac{1}{Q}V\beta f^2(k)E\left(Xu''\right)+\frac{1}{Q}U\beta f^2(k)E\left(mu''\right)\right]\xi=0$$

The eigenvalues are $\xi_1=0$ and $\xi_2=\dfrac{1}{Q}V\beta f^2(k)E\left(Xu''\right)+\dfrac{1}{Q}U\beta f^2(k)E\left(mu''\right)$.

Let $O=U\beta f^2(k)E\left(mu''\right)+V\beta f^2(k)E\left(Xu''\right)$, we need to prove $O\geq 0$. According to Cauchy-Schwarz inequality, we have

$$Q-O=\left\{\beta Eu''E\left(X^2u''\right)-2PEu''E\left(Xu''\right)+\frac{1}{\beta}P^2Eu''Eu''\right\}\left[1+Pf'\left(k\right)\right]^2 f^2\left(k\right)$$

$$\geq\left\{\beta E\left(Xu''\right)E\left(Xu''\right)-2PEu''E\left(Xu''\right)+\frac{1}{\beta}P^2Eu''Eu''\right\}\left[1+Pf'\left(k\right)\right]^2 f^2\left(k\right)$$

$$=\left[\sqrt{\beta}E\left(Xu''\right)-\frac{1}{\sqrt{\beta}}PEu''\right]^2\left[1+Pf'\left(k\right)\right]^2 f^2\left(k\right)\geq 0$$

which means $\xi_2<1$.